Time-reversed flows

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Under the assumption that the local acceleration and inertial terms are to be neglected in the equation of motion, a comparison is made between the unsteady-state movement of a suspension of particles and the same problem in reversed (or negative) time. The formalized discussion confirms Bretherton's (1962) conclusion that, if in a steady unidirectional shear flow at small Reynolds number a rotationally symmetric particle twice assumes a position with its axis of rotation in the plane perpendicular to the flow, its orbit must be periodic. The reversibility phenomena observed by S. G. Mason (1963) in dilute and concentrated suspensions are explained as well.

1. Introduction

Bretherton (1962) considered the movement of a fluid carrying a body of revolution through a cylindrical channel (not necessarily circular cross-section). From an argument in which the flow is compared with the same situation in reverse time, he concluded that, if a reversing mechanism exists such that the particle has its axis of symmetry in the plane orthogonal to the tube axis at two different points along the axis, the particle must follow a periodic trajectory.

Mason (1963) observed several interesting reversibility phenomena. In a clear, viscous material contained between two concentric glass cylinders he drew several letters with a dye. He first slowly rotated one of the cylinders until, to the eye, the dye appeared to be mixed with the clear solution which had originally surrounded the letters; after stopping all movement, he slowly rotated the same cylinder in the opposite direction until the letters again appeared, blurred only in a minor way—apparently by diffusion. In a second experiment in the same geometry he observed a suspension of cylindrical particles which had been given a specific orientation. He again rotated one of the bounding walls until the cylinders appeared to be in a random arrangement, stopped, and reversed the rotation until the particles resumed their initial orientation.

The argument of Bretherton and the experiments of Mason suggested the following treatment of time-reversed flows of suspensions of particles.

2. Unsteady-state flow of a suspension of particles

In this section we ask merely what equations and boundary conditions must be satisfied in the movement of a suspension of particles in a given geometry. We restrict attention to incompressible flows in which inertial and local acceleration effects may be neglected in the equation of motion.

The equation of continuity reduces for an incompressible material to

$$(\nabla . \mathbf{v}) = 0, \tag{1}$$

where $\mathbf{v}$ represents the velocity vector.

The stress equation of motion becomes under the restrictions mentioned above

$$0 = (\nabla . \mathbf{t}) + \rho \mathbf{f}. \tag{2}$$

Here $\mathbf{t}$ is the stress, a second-order tensor, given by

$$\mathbf{t} = -p\mathbf{I} + \boldsymbol{\tau}, \tag{3}$$

where p, the pressure, $= -\frac{1}{3}$ trace $(\mathbf{t})$, $\mathbf{I}$ is the unit tensor, $\boldsymbol{\tau}$ is the extra stress, a second-order tensor sometimes referred to as the viscous portion of the stress, $\mathbf{f}$ is the external body force vector and ρ is the density. If we restrict ourselves to cases where the external body-force vector can be represented in terms of a potential

$$\mathbf{f} = -\nabla \phi, \tag{4}$$

and the fluid behaviour is Newtonian† with viscosity μ,

$$\boldsymbol{\tau} = \mu[\nabla \mathbf{v} + (\nabla \mathbf{v})^t].\ddagger \tag{5}$$

Equation (2) becomes

$$0 = -\nabla P + (\nabla . \boldsymbol{\tau}) = -\nabla P + \mu \nabla^2 \mathbf{v}, \tag{6}$$

where

$$P = p + \rho \phi. \tag{7}$$

The force $\mathbf{F}$ exerted on a particle carried along in a moving fluid stream may be expressed as the sum of the contact forces and the external body forces:

$$\mathbf{F} = \int_{S_p} (\mathbf{t}_f . \mathbf{n}) \, dS + \int_{V_p} \rho_p \mathbf{f} \, dV. \tag{9}$$

The quantities S_p and V_p are respectively the closed bounding surface and volume of the particle; $\mathbf{n}$ is the unit vector, normally and outwardly directed with respect to the closed surface S_p; ρ_p is the density of the particle; $\mathbf{t}_f$ is the stress in the fluid phase. Using equations (3), (4) and the divergence theorem and observing that in the absence of any surface effects

$$\mathbf{t}_f . \mathbf{n} = \mathbf{t}_p . \mathbf{n}, \tag{10}$$

† The argument presented here might be extended to fluids whose behaviour is other than Newtonian; the only requirement is that the expression for $\boldsymbol{\tau}$ be such that the time-reversed stress be the negative of the stress in real time (see equation (30)). The argument would not hold for the simple fluid of Noll (1958) (see Coleman & Noll 1961) as can be readily seen by examining a special case, the Reiner–Rivlin fluid (see Reiner 1945 and Rivlin 1947):

$$\boldsymbol{\tau} = \alpha \mathbf{d} + \beta(\mathbf{d} . \mathbf{d}), \tag{8}$$

where

$$\alpha = \alpha(II, III), \quad \beta = \beta(II, III),$$
$$II = \mathbf{d} : \mathbf{d}, \quad III = \det(\mathbf{d}).$$

On the other hand, it could be immediately extended to the class of generalized Newtonian fluids ($\beta = 0$ in equation (8)), but it is not clear that this form of constitutive equation represents the behaviour of any non-Newtonian fluid. A generalized Newtonian fluid would not for example exhibit normal-stress phenomena (see Markovitz & Brown 1963).

‡ By $(\nabla \mathbf{v})^t$ we mean the transpose of $\nabla \mathbf{v}$.

where $\mathbf{t}_p$ is the stress within the particle; we may rewrite equation (9) as

$$\mathbf{F} = \int_{S_p} [\mathbf{t}_p . \mathbf{n}]\, dS - \int_{V_p} \nabla(\rho_p\, \phi)\, dV \tag{11}$$

$$= \int_{V_p} [\nabla . (-P_p\, \mathbf{I} + \boldsymbol{\tau}_p)]\, dV. \tag{12}$$

The balance of momentum (see Truesdell & Toupin 1960) applied to a particle requires

$$\mathbf{F} = \frac{d}{dt}\int_{V_p} \rho_p\, \dot{\mathbf{p}}\, dV. \tag{13}$$

We denote by $\mathbf{p}$ the position vector with respect to some frame fixed in space of any point fixed in the particle and by $\dot{\mathbf{p}}$ its time rate of change. Since equation (6) must be satisfied everywhere (in both phases), we have from equations (12) and (13)

$$0 \doteq \frac{d}{dt}\int_{V_p} \rho_p\, \dot{\mathbf{p}}\, dV. \tag{14}$$

Let us next look at the balance of moment of momentum (see Truesdell & Toupin 1960) applied to a particle being carried along with the fluid. The torque $\mathbf{L}$ on such a body is the sum of the moments of the contact forces and of the external body force,

$$\mathbf{L} = \int_{S_p} [\mathbf{p} \times (\mathbf{t}_f . \mathbf{n})]\, dS + \int_{V_p} (\mathbf{p} \times \mathbf{f})\, \rho_p\, dV. \tag{15}$$

By a series of arguments which depend upon equation (10), the divergence theorem, the fact that

$$\nabla \mathbf{p} = \mathbf{I}, \tag{16}$$

that $\boldsymbol{\tau}$ is a symmetric tensor, that the density of the fluid is nearly that of the particle, and that equation (6) applies everywhere, we may rearrange equation (15) to give

$$\mathbf{L} = \int_{S_p} [\mathbf{p} \times (\mathbf{t}_p . \mathbf{n})]\, dS - \int_{V_p} [\mathbf{p} \times \nabla(\rho_p\, \phi)]\, dV \tag{17}$$

$$= \int_{S_p} [\mathbf{p} \times (\mathbf{t}_p . \mathbf{n})]\, dS - \int_{S_p} [\rho_p\, \phi(\mathbf{p} \times \mathbf{n})]\, dS \tag{18}$$

$$= \int_{S_p} [\mathbf{p} \times (\{-P_p\, \mathbf{I} + \boldsymbol{\tau}_p\} . \mathbf{n})]\, dS \tag{19}$$

$$= \int_{V_p} [\mathbf{p} \times (\nabla . \{-P_p\, \mathbf{I} + \boldsymbol{\tau}_p\})]\, dV = 0. \tag{20}$$

The moment-of-momentum balance applied to the particle reduces to

$$\mathbf{L} = \frac{d}{dt}\int_{V_p} [(\mathbf{p} \times \dot{\mathbf{p}})\, \rho_p]\, dV = 0. \tag{21}$$

The boundary conditions which must be satisfied in a problem of this sort are of three types: velocity boundary conditions on a portion of the bounding surface of the system, S_v, stress boundary conditions on a portion of the bounding surface of the system, S_t, and velocity boundary conditions on the surfaces, S_p, of any suspended particles. These may be represented respectively by

$$S_v: \mathbf{v} = \mathbf{V} = \mathbf{V}(\mathbf{p}, t), \tag{22}$$

$$S_t: \mathbf{t}.\mathbf{n} = \mathbf{T} = \mathbf{T}(\mathbf{p}, t), \tag{23}$$

$$S_p: \mathbf{v} = \dot{\mathbf{p}}. \tag{24}$$

The quantities $\mathbf{V}$ and $\mathbf{T}$ are known vector functions of position on the surface and of time. By $\mathbf{n}$ in equation (23) we mean the unit vector normal to the surface S_t and directed outward into the suspension enclosed within the surface $S = S_v + S_t$.

As initial conditions we require that at time $t = 0$ the velocity be known everywhere

$$t = 0: \mathbf{v} = \mathbf{g} = \mathbf{g}(\mathbf{p}), \tag{25}$$

and that the location and orientation of all particles be known.

In summary, the correct description of the unsteady-state movement of a suspension of particles from some initial distribution and orientation must satisfy equations (1) and (6) everywhere within the fluid, equations (14) and (21) for each particle, boundary conditions (22)–(24), initial condition (25), and a given initial distribution and orientation of all particles.

3. Unsteady-state flow of a suspension of particles in reverse time

A time-reversed system can be thought of as a ciné-film of a real system run backwards; starting with some arbitrary time $t = 0$, it shows the development of the flow in reverse. Time in the time-reversed system, $\bar{t}$, is the negative of real time t (barred quantities will refer to the time-reversed system), i.e.

$$\bar{t} = -t, \tag{26}$$

and the time-reversed velocity vector $\bar{\mathbf{v}}$ is the negative of the real velocity vector. In terms of the position vector $\mathbf{p}$,

$$\bar{\mathbf{v}} = \frac{d\mathbf{p}}{d\bar{t}} = -\frac{d\mathbf{p}}{dt} = -\mathbf{v}. \tag{27}$$

In § 2 we discussed the general problem of the movement of a suspension of particles under conditions such that inertial and local acceleration effects are negligible in the equation of motion. Looking only at the equation of motion, we see that in general the result for the time-reversed system,

$$-\nabla P - \mu \nabla^2 \bar{v} = 0, \tag{28}$$

is not the same as for the real system, equation (6). This means merely that the pressure gradient in the real system is the negative of that in the corresponding time-reversed system; this checks with our idea of a time-reversed system as a ciné-film run backwards.

On the other hand, if we restrict our attention to problems in which velocity is specified everywhere on the bounding surface of the system, we do not wish to solve equations (6) and (28), but rather

$$[\nabla \times \nabla^2 \mathbf{v}] = \mathbf{0}, \quad [\nabla \times \nabla^2 \bar{\mathbf{v}}] = \mathbf{0}. \tag{29}$$

Equations (29) are the basis for the discussion in §§ 4 and 5.

4. Trajectory of a rotationally-symmetric particle carried in a moving fluid

Let us consider a fluid which moves through a straight conduit of constant cross-section (but arbitrary shape) and which carries a single, rotationally-symmetric particle. We take as time $t = 0$ an instant at which the axis of symmetry of the particle lies in a plane normal to the axis of flow, so that the description of the initial position of the particle for forward motion is the same as that for time-reversed motion in a co-ordinate system reflected in the plane normal to the axis of flow and passing through the axis of symmetry of the particle. We consider the velocity very far upstream and downstream to be specified. From equations (1), (29), (14) and (21), from the boundary conditions (22) and (24), and from the initial conditions (25), it is obvious that the equations and boundary conditions for the time-reversed problem are exactly the same as for the corresponding real problem. This means that, if a reversing mechanism exists such that the axis of symmetry of the particle twice lies in a plane normal to the axis of flow, the trajectory of the particle will be periodic. It should be emphasized that the assumptions leading to this conclusion are that inertial and local acceleration effects can be neglected in the equation of motion and that the suspending fluid is Newtonian (see footnote on p. 626).

The above conclusion was correctly stated by Bretherton (1962) (see his § 2.2); but the argument as he stated it depended upon the pressure in the time-reversed system being the negative of the pressure in the real system at the corresponding negative time (see his § 2.1), which is incorrect. We saw in § 3 that the gradient of pressure in the time-reversed system was the negative of that in the corresponding real flow at the corresponding negative time.

5. Explanation of the reversibility phenomena observed by S. G. Mason

In § 1 two observations by Mason (1963) of 'reversible' flows were outlined. We attempt to explain his observations below.

The essential feature of Mason's observations was that, given an initial particle distribution and orientation in a fluid, he developed an unsteady-state flow. After a given period of time he stopped the first flow and began another; all velocities on the bounding surfaces of the system were the negative of those in the first flow. He found that the trajectories of the particles in the second experiment were seemingly those which one might observe if a ciné-film of the first experiment were run backwards, which suggests the explanation.

Let us compare the equations, boundary conditions, and initial conditions to be satisfied by the second flow and compare them with those of the time-reversed first flow. From equations (1), (29), (14), (21), (22) and (24) we see that the equations and boundary conditions are the same. The initial velocity distribution is the same in both cases—the velocity is zero everywhere. Further, the initial particle distributions and orientations are the same for both flows. This shows us that the velocity distribution as a function of time in the second motion is the reverse-chronological velocity distribution for the first motion, exactly what Mason observed.

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